Linearity and the Doctrinal Paradox

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Abstract

Suppose that opinions over a set of atomic propositions are combined into an opinion over some issue φ through a function f, and that opinions of different agents about the same atomic proposition are aggregated according to some function g. We examine aggreganetion of opinions that employs function f first and then function g ($g \circ f$) and aggregation of opinions that employs function g first and then function f ($f \circ g$). We provide sufficient conditions for the order in which f and g are combined not to matter, or for f and g to "aggregate," and relate our results to the doctrinal paradox. We conclude with a conjecture that provides a characterization of aggregation.

Keywords: Aggregation of opinions, Doctrinal Paradox.

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1 Introduction

The aggregation of binary judgement is well known to be problematic. This difficulty is probably best illustrated by the so called 'doctrinal paradox' that is described in Section 3 below, and has received a lot of attention in the literature (see Guilbaud (1966), Kornhauser and Sager (1986, 1993), Kornhauser (1992), Chapman (1998, 2003), Pettit (2001), List and Pettit (2002, 2004), Pauly and van Hees (2003), Bovens and Rabinowicz (2004a, 2004b), Dietrich (2006, 2007), List (2004, 2005), Dietrich and List (2005), Nehring and Puppe (2005), and Dokow and Holzman (2005)).

Briefly stated, the difficulty is that "propositionwise majority voting over multiple interconnected propositions may lead to inconsistent collective sets of judgments on these propositions" (List and Pettit, 2004, p. 2). Namely, aggregation of agents' judgments over single propositions using majority rule followed by combination of these aggregated judgments in order to decide on the truth value of a compound proposition, may produce a different result from the one that is obtained when each agent decides on the truth value of the compound proposition on its own, and then majority rule is used to decide among these compound judgments.

In this note we examine how general is the type of difficulty that is described in the doctrinal paradox. In our setup, agents may hold opinions that can take any value in the interval $[0, 1]$ rather than the usual dichotomous binary judgment in the literature about the aggregation of judgment. Suppose that opinions over a set of atomic propositions are combined into an opinion over some issue φ through a function f, and that opinions of different agents about the same atomic proposition are aggregated according to some function g. It could be readily verified that there is a wide set of functions f and g for which an aggregation of opinions that employs function f first and then function g $(g \circ f)$ and aggregation of opinions that employs function g first and then function f $(f \circ g)$ produces a different final opinion about the issue φ. However, in *some* (but not *all)* cases linearity may guarantee that $(f \circ g) = (g \circ f)$. We present the necessary and sufficient condition for linearity and robustness against the paradox.

2 Setup

Let $N = \{1, ..., n\}$ denote a (finite) set of agents, and $K = \{1, ..., k\}$ denote a (finite) set of "atomic propositions." Each agent $i \in N$ has an opinion $o_p^i \in [0, 1]$ about each atomic proposition $p \in K$ that expresses the agent's degree of self-persuasion about the truth value of the proposition. The objective is to aggregate the profile of agents' opinions about the different atomic propositions into a joint aggregate opinion about a certain issue denoted φ for final.

An "atomic proposition aggregator" function $f : [0,1]^K \to [0,1]$ is a mapping from a vector of opinions about the atomic propositions into an aggregate opinion about φ, and an "agent aggregator" function $g : [0,1]^N \to [0,1]$ is a mapping from a vector of agents' opinions about a single atomic proposition or about φ into an aggregate opinion about the truth value of, or the group's degree of self persuasion about, this atomic proposition or φ, respectively.

The two functions f and g can be combined to produce a joint aggregate opinion about φ in either one of the following two ways:

- **First** f **then** g ($g \circ f$). First apply f to each agent's opinions about the atomic propositions to produce each agent's opinion about φ, denoted φ^i, respectively; then apply g to the agents' aggregate opinions $\left(\varphi^1, ..., \varphi^N\right)$ to produce a joint aggregate opinion about φ.

- **First** g **then** f ($f \circ g$). First, for each atomic proposition $p \in K$, apply g to the agents' opinions about this proposition $\left(o_p^1, ..., o_p^N\right)$ to produce an aggregate opinion about this atomic proposition, denoted o_p; then apply f to the aggregate opinions about the propositions $(o_1, ..., o_K)$ to produce a joint aggregate opinion about φ.

  If the joint aggregate opinion about φ is independent of the order in which the two functions f and g are combined, then we say that *f and g aggregate.*

3 Aggregation of Opinions

The question we pose is for what functions f and g do the two methods of aggregation described above produce the same opinion about φ. The following example, which has come to be known as the 'doctrinal paradox' or 'discursive dilemma' illustrates that for two functions f and g that arise naturally in the context of opinion or judgment aggregation these two methods of aggregation produce different results.

Example. (Kornhauser and Sager, 1993, p. 11) Suppose that a three-judge court has to make a decision on whether a defendant is liable for breach of contract. Suppose that according to the prevailing legal regime, the defendant is liable if and only if the contract is valid (atomic proposition 1), and the contract was breached (atomic proposition 2). Hence, according to our notation, the function f that maps opinions about the two atomic propositions into a judgment about liability produces the value 1 (the defendant is liable) if and only if both its arguments are 1. In all other cases, the function f produces the value 0 (the defendant is not liable).

Suppose that the judges' opinions about the two atomic propositions are as described in the table below:

Judge	*valid contract*	*breach*	*defendant liable*
1	*yes* $(o_1^1 = 1)$	*yes* $(o_2^1 = 1)$	*yes* $(\varphi^1 = 1)$
2	*yes* $(o_1^2 = 1)$	*no* $(o_2^2 = 0)$	*no* $(\varphi^2 = 0)$
3	*no* $(o_1^3 = 0)$	*yes* $(o_2^3 = 1)$	*no* $(\varphi^3 = 0)$
median judge	*yes* $(o_1 = 1)$	*yes* $(o_2 = 1)$	*yes\no*

Suppose that the function g aggregates the judges' opinions according to majority rule, that is g assigns to each atomic proposition the judges' median opinion. Observe that if the aggregation is done in the 'first f then g' $(g \circ f)$ method, then the defendant is found not liable, but if the aggregation is done in the 'first g then f' $(f \circ g)$ method then the liable is found liable.

The next Lemma describes a sufficient condition for aggregation. An aggregation function $f : [0,1]^n \to [0,1]$ is said to be linear if $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ and $f(c\mathbf{v}) = cf(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v} \in [0,1]^n$ and all scalars $c \in [0,1]$. Note that a linear function f is of the form $f(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$ for some fixed scalars $c_1, \ldots, c_n$ and all $x_1, \ldots, x_n \in [0,1]$, and that since $[0,1]^n$ contains the standard basis $\{(1,0,\ldots,0), \ldots, (0,\ldots,0,1)\}$ of $\mathbb{R}^n$, if f is linear, then there exists a unique linear map $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ that extends f to $\mathbb{R}^n$.

Lemma 1. If both f and g are linear functions, then they aggregate for every profile of agents' opinions.

Proof. Suppose that f and g are linear functions. Hence, there exist two vectors $(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k$ and $(\beta_1, ..., \beta_n) \in \mathbb{R}^n$ such that $f(o_1, ..., o_k) = \sum_{p=1}^{k} \alpha_p o_p$ for every vector $(o_1, ..., o_k) \in [0,1]^k$ and $g(\varphi_1, ..., \varphi_n) = \sum_{i=1}^{n} \beta_i \varphi_i$ for every vector $(\varphi_1, ..., \varphi_n) \in [0,1]^n$.

Due to linearity of f and g :

$$f(\{o_p^i\}_{p=1,\ldots,k}) = \sum_{p=1,\ldots,k} \alpha_p o_p^i \tag{1}$$

for some vector $(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k$ for every vector $\{o_p^i\}_{p=1,...,k} \in [0,1]^k$ and

$$g(\{o_p^i\}_{i=1,...,n}) = \sum_{i=1,...,n} \beta_i o_p^i \tag{2}$$

for some vector $(\beta_1, ..., \beta_n) \in \mathbb{R}^n$ for every vector $\{o_p^i\}_{i=1,...,n} \in [0,1]^n$.

We have to show that:

$$\begin{aligned} A &\equiv f\left(g(\{o_1^i\}_{i=1,...,n}), ..., g(\{o_k^i\}_{i=1,...,n})\right) \\ &= g(f(\{o_p^1\}_{t=1,...,k}), ..., f(\{o_p^n\}_{p=1,...,k})) \\ &\equiv B \end{aligned}$$

By definition of f and g (eq. 1 and 2), A is equal to:

$$\begin{aligned} A &\equiv \alpha_1 g(\{o_1^i\}_{i=1,...,n}) + \cdots + \alpha_k g(\{o_k^i\}_{i=1,...,n}) \\ &= \sum_{p=1..k} \alpha_p \sum_{i=1..n} \beta_i o_p^i \\ &= \sum_{p=1..k} \sum_{i=1..n} \alpha_p \beta_i o_p^i \end{aligned} \tag{3}$$

and B is equal to:

$$\begin{aligned} B &\equiv \beta_1 f(\{o_p^1\}_{p=1,...,k}) + \cdots + \beta_n f(\{o_p^n\}_{p=1,...,k}) \\ &= \sum_{p=1..n} \beta_p \sum_{i=1..k} \alpha_i o_p^i \\ &= \sum_{p=1..k} \sum_{i=1..n} \alpha_p \beta_i o_p^i \end{aligned} \tag{4}$$

hence, $A = B$, which completes the proof. ■

Lemma 1 provides a sufficient condition for aggregation, or for robustness against the phenomenon described by the doctrinal paradox. The following proposition describes a necessary and sufficient condition for aggregation.

Proposition 1. Let $f : [0,1]^n \to [0,1]$ and let $m \geq 2$. Then f aggregates with every linear function $g : [0,1]^m \to [0,1]$ if and only if f is linear.

Proof. ($\Longleftarrow$) Let $f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$ for some scalars $c_1, \ldots, c_n$ and for all

$x_1, \ldots, x_n$ in $[0,1]$. Thus for any $k \times n$ matrix $B = (b_{ij})$ over $[0,1]$, where $k \geq 1$, we have

$$B \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix} = \begin{pmatrix} f(b_{11}, & \ldots & , b_{1n}) \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots \\ f(b_{k1}, & \ldots & , b_{kn}) \end{pmatrix}.$$

Let $F = \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix}$.

Similarly, if g is a linear function, there exists a row $G \in \mathbb{R}^m$ so that the action of g on the columns of a $m \times k$ matrix B is given by GB.

Hence, if we apply first f on the matrix A, then g, we obtain $G(AF)$; if we apply first g, then f, we obtain $(GA)F$ and the two results are equal by associativity of matrix multiplication.

($\Longrightarrow$) Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be m row vectors in $[[0,1]^n$. By assumption, we have for any scalars $c_1, \ldots, c_m \in [0,1]$ that $f(c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m) = c_1 f(\mathbf{v}_1) + \cdots + c_m f(\mathbf{v}_m)$. For $c_1 = \cdots = c_m = 0$ we obtain that $f(\mathbf{0}) = 0$. Let c by a scalar in $[0,1]$, and set $c_1 = c$ and $c_i = 0$ for $i > 1$. We see that $f(c\mathbf{v}_1) = cf(\mathbf{v}_1)$. Now set $c_1 = c_2 = 1$ and $c_i = 0$ for $i > 2$ to obtain that $f(\mathbf{v}_1) + \mathbf{v}_2) = f(\mathbf{v}_1) + f(\mathbf{v}_2)$. We conclude that f is linear. ■

One may wonder whether there exists an aggregation function that aggregates with any other function. The answer to this question is affirmative, as we show in the next proposition; We show that only projections (that is *not just any* dictatorial function) aggregate:

Proposition 2. Let $m, n \geq 2$ be two integers. Then the functions $f : [0,1]^n \to [0,1]$ that aggregate with every function $g : [0,1]^m \to [0,1]$ are precisely the projections $p_i : (x_1 \ldots, x_n) \mapsto x_i$.

Proof. By direct verification we obtain that the projections aggregate with any function.

For the converse, suppose that f aggregates with every function $g : [0,1]^m \to [0,1]$. By Proposition 1, f is linear:

$$f : (x_1, \ldots, x_n) \to c_1x_1 + \cdots + c_nx_n.$$

Since f aggregates with the function x_1^2, using the $m \times n$ matrix A with all its rows equal to $(x_1, \ldots, x_n)$, we obtain that $c_1x_1^2 + \cdots + c_nx_n^2 = (c_1x_1 + \cdots + c_nx_n)^2$ for any scalars $x_1, \ldots, x_n$ in $[0, 1]$. Plug in $x_1 = 1$ and $x_i = 0$ for $i > 1$ to get that $c_1^2 = c_1$, thus $c_1 = 0$ or 1. Similarly, $c_i = 0, 1$ for all i. If, e.g., $c_1 = c_2 = 1$, then we plug in $x_1 = x_2 = 1$ and $x_i = 0$ for $i > 0$ to obtain the contradiction $2 = 4$. Thus at most one of the coefficients c_i is 1. Hence either f is a projection or the constant zero function. But the zero function does not aggregate with a constant nonzero function. We conclude that f is a projection.[1] ■

4 Concluding Remarks

This note has suggested a setting in which judgments are expressed over the interval $[0, 1]$ rather than over the set $\{0, 1\}$, hence, unlike much of the rest of the literature about judgment aggregation, agents in our setup are able to express the intensity of their judgment.

We have shown under what setting aggregation is "path-independent", and examined the case where the "path-aggregator" is linear. Our results generalize the phenomenon described by the doctrinal paradox to a large family of aggregators. It is shown that the linearity of a single aggregator (that is only f or g) may not be robust against the paradox. Our results also provide a characterization of a set of aggregators that are not vulnerable to the doctrinal paradox.

We conclude with a conjecture that provided the motivation for this note, which we have been unable to prove, and that seems like a surprisingly difficult question. Let a function $f : [0, 1]^n \to [0, 1]$, $n \geq 2$, be called *nondictatorial* if it depends on at least two of its arguments, and let a function $f : [0, 1]^n \to [0, 1]$, $n \geq 2$, be called a *power function* if it is of the form

$$f : (x_1, \ldots, x_n) = c_0 x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}.$$

for some fixed scalars $c_0, c_1, \ldots, c_n$ and all $x_1, \ldots, x_n \in [0, 1]$.

Conjecture. A pair of non dictatorial functions $f : [0, 1]^n \to [0, 1]$ and $g : [0, 1]^m \to [0, 1]$, $n, m \geq 2$ aggregate if and only if f and g are either linear up to a constant, or "power functions."

Example. Let two agents' opinions about two atomic propositions be given by the following

[1] In the proof we use just one test function, namely, the function x_1^2, to obtain that a linear function that aggregates with x_1^2 is a projection.

matrix:

$$O = \begin{pmatrix} o_{11} & o_{12} \\ o_{21} & o_{22} \end{pmatrix}$$

Two functions $f(x_1, x_2) = c_0 + c_1 x_1 + c_2 x_2$ and $g(x_1, x_2) = d_0 + d_1 x_1 + d_2 x_2$ aggregate if $c_0 + d_0 (c_1 + c_2) = d_0 + c_0 (d_1 + d_2)$, and two functions $f(x_1, x_2) = c_0 x_1^{c_1} x_2^{c_2}$ and $g(x_1, x_2) = d_0 x_1^{d_1} x_2^{d_2}$ aggregate if $d_0 c_0^{d_1+d_2} = c_0 d_0^{c_1+c_2}$. We conjecture that these are the only pairs of function that aggregate, but are unable to prove it.

If our conjecture is correct, then aggregation is very special, and phenomena like those described in the doctrinal paradox, are the rule, rather than a curious exception.

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